NOTES AND CORRESPONDENCE

Further Comments on
"A Comparison of the Equivalent Potential Temperature and the Static Energy"

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1. Introduction

The note by Madden and Robitaille (1970) (abbreviated MR 70) and the subsequent comments (Levine, 1972) and reply by the authors (1972) deserve further clarification with the increasing use of static energy in convective parameterization models (Arakawa and Schubert, 1974; Fraedrich, 1973; Yanai et al., 1973; Ogura and Cho, 1973). The original note was correct in stating that the available potential energy which can be converted into kinetic energy is related to the different parcel equilibrium height obtained by using static energy instead of equivalent potential temperature. This difference has further theoretical significance.

2. Reversible water saturation adiabat

No thermodynamic reference process can be more than an approximation to an atmospheric process, but undilute parcel ascent paths without freezing will normally lie between a reversible water saturation adiabat and a pseudoadiabat, depending on the fractional fallout of liquid water. The exact equation for the reversible water saturation adiabat (Fjeldstad, 1925; Brunt, 1934) is

$$dS = 0 = (C_p + rC)\frac{dT}{T} + d\left(\frac{Lr_s}{T}\right) - R_d\frac{dp_d}{p_d}, \qquad (1)$$

where dS is the entropy change of the system containing 1 gm of air and r gm of water; C_p, R_d, p_d are the specific heat, gas constant and partial pressure of dry air; and r_s is the saturation mixing ratio and C the specific heat of liquid water.

The saturation pseudoadiabat for an open system (see Van Mieghem, 1951) is similar to Eq. (1), but with r_sC replacing rC in the first term. The neglect of this same term gives the Rossby approximation (Rossby, 1932; Wurtele and Finke, 1961) to the pseudoadiabats (as used by MR 70 and Levine). For the purpose here of a theoretical discussion, it is most

convenient to use Eq. (1) since this represents a closed system of fixed mass.

Several exact manipulations of Eq. (1) are informative. Using the Clausius-Clapyron equation, one can derive

$$\frac{dS}{1+r} = 0 = C_p'\frac{dT}{T} + \frac{d(Lq_s)}{T} - R'\frac{dp}{p}, \qquad (2)$$

with the definitions

$$\left.\begin{array}{l} (1+r)C_p' = C_p + rC \\ (1+r)q_s = r_s \\ (1+r)R' = R_d + r_sR_v \end{array}\right\},$$

where R_v is the gas constant for water vapor and p the total pressure of the mixture.

Eqs. (1) or (2) define an isentropic path. One may integrate (1) or (2) (or more commonly the equivalent pseudoadiabat) to define a path of constant saturation equivalent potential temperature (θ_{es}) by setting $\theta = \theta_{es}$ as $q_s \rightarrow 0$. In a (fictional) atmosphere at rest with this vertical temperature, vapor and liquid water structure, and density ρ', one can use the hydrostatic equation to define a z' (following Green et al., 1966):

$$dp = -\rho' g dz'. \qquad (3)$$

Combining Eqs. (2), (3) and the gas law for the model atmosphere gives

$$\frac{TdS}{1+r} = 0 = C_p'dT + d(Lq_s) + gdz'. \qquad (4)$$

This can be integrated, taking C_p, C, g as constants, to give a conserved function,

$$h_\phi = C_p'T + Lq_s + gz', \qquad (5)$$

where z' will be taken as zero at 1000 mb; h_ϕ is a constant along a reversible adiabat and is uniquely related to S, and a θ_{es} defined by integrating Eqs. (1) or (2). If a dry atmosphere is taken as a special case of Eq. (5), then h_ϕ becomes $C_p\theta$. For our model atmosphere,

as $q_s \rightarrow 0$, $h_\phi \rightarrow C_p'T + gz'$. On the same adiabat as $q_s \rightarrow 0$, one can define

$$\theta_{es} = \theta = T + gz_d/C_p,$$

where z_d is the height above 1000 mb in the dry atmosphere with constant potential temperature θ. Thus,

$$C_p'\theta_{es} - h_\phi = g\left(\frac{C_p'z_d}{C_p} - z'\right)_{q_s \rightarrow 0}.$$

This difference is a constant for a given reversible adiabat. Whether h_ϕ, θ_{es} or S are used to label an adiabat, they all give the same equilibrium level.

3. Static energy and parcel equilibrium heights

The function h_ϕ [Eq. (5)] clearly bears a relationship to a saturation static energy. We shall define a corresponding saturation static energy h_s

$$h_s = C_p'T + Lq_s + gz, \qquad (6)$$

where z is the height above 1000 mb in an environmental atmosphere, which in general will not have the temperature structure of the reversible water saturation adiabat. Eq. (6) has been defined to correspond exactly with Eq. (5), except that h_s differs from h_ϕ by the term $g(z' - z)$. This difference is the maximum available kinetic energy per unit mass of an infinitesimal parcel ascending from $z = 0$ along the reversible adiabat, in the stationary environmental atmosphere (see Green et al., 1966). Clearly, both the environment and parcel defined by Eqs. (5) and (6) as having the same constant water mixing ratio r (implicit in C_p') are fictional, and the definitions h_ϕ,
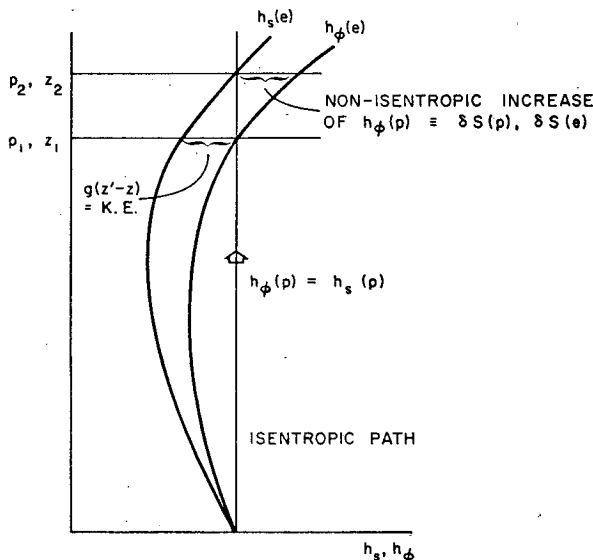

FIG. 1. Equilibrium heights for an ascending parcel: z_1 based on isentropic ascent to thermal equilibrium; z_2 based on using environmental static energy curve or parcel ascent with total internal dissipation of kinetic energy.

h_s will only be of practical use if one approximates $C_p' \approx C_p$. For a dry atmosphere and parcel, this difficulty does not arise.

The parcel will gain kinetic energy on ascent if $z' > z$, so that when an ascending parcel reaches thermal equilibrium with the environment (i.e., both parcel and environment have the same value of h_ϕ, θ_{es} or S), $h_s < h_\phi$. Thus, if the saturation static energy is used to define an equilibrium height, this will be overestimated (see Fig. 1). However, this higher equilibrium height does have a theoretical meaning: one can show that if the kinetic energy $g(z' - z)$ is hypothetically dissipated and converted back into the enthalpy of the parcel, then h_ϕ (θ_{es} and S), being no longer conserved, are increased so that the new thermal equilibrium height is given by this higher level.

Fig. 1 is a sketch (not to scale) of a parcel $h_\phi(p)$, and environment $h_\phi(e)$ and $h_s(e)$, illustrating these two equilibrium heights z_1 and z_2 [typically $\delta p = (p_1 - p_2) \approx 10$ mb]. At z_1, the thermal equilibrium height after isentropic ascent, the maximum kinetic energy is given by

$$\text{K.E.} = g(z' - z) = [h_\phi(e) - h_s(e)]_{z_1} = \delta h_s(e), \quad (7)$$

where $\delta h_s(e)$ is the difference in $h_s(e)$ from z_1 to z_2. Since entropy is a state function, we can use Eq. (2) (not set equal to zero) to define the entropy difference $\delta S(e)$ for unit mass of the environment corresponding to $\delta h_s(e)$ between z_1 and z_2; i.e.,

$$T(e)\delta S(e) = C_p'\delta T(e) + \delta[Lq_s(e)] - R'T(e)\delta p/p.$$

Using the hydrostatic relation for the small difference in pressure gives

$$T(e)\delta S(e) = \delta h_s(e). \qquad (8)$$

For the parcel, the total dissipation of the available kinetic energy to heat δQ will increase the parcel entropy

$$\delta S(p) = \frac{\delta Q}{T(p)} = \frac{g(z' - z)}{T(p)} = \frac{\delta h_s(e)}{T(p)}.$$

Since near thermal equilibrium $T(p) \approx T(e)$, we find

$$T(e)\delta S(p) = \delta h_s(e). \qquad (9)$$

It follows from Eqs. (8) and (9) that the entropy increase of the parcel equals the entropy difference of the environment between z_1 and z_2. Thus, z_2 is the new thermal equilibrium height since h_ϕ, θ_{es}, S, T are uniquely related at a given pressure for the chosen model atmosphere, and reference parcel path.

For a dry atmosphere and parcel, the analysis above is more straightforward, since $\delta S = C_p\delta\theta/\theta$, and h_ϕ becomes $C_p\theta$.

Using the figures from MR 70, the thickness difference $g(z' - z)$ from 1000 to 180 mb between the 348.6K saturation pseudoadiabat and the mean sounding is 95 m. Thus from Eq. (7) we have $\delta h_s(e) = 0.22$ cal

gm^{-1}. This corresponds in MR 70 to an 11-mb change in parcel top height; the authors reported a 9-mb difference. However, the strict thermal equilibrium height is given by the level where the parcel ascent curve intersects the environmental θ_{es} curve [or $h_\phi(e)$ curve], which in this case is about 2 mb lower than the level given in MR 70 based on the environmental θ_e curve.

4. Derivation of static energy from the first law

This neglect of the kinetic energy generation by the use of the static energy can be illustrated more generally, starting from the first law of thermodynamics:

$$C_p\frac{dT}{dt}-\alpha\frac{dp}{dt}=\dot{Q}, \tag{10}$$

where $\dot{Q}$ is a non-frictional heating rate, including condensation, and α is specific volume. The specific heat will be approximated by the value for dry air. With the approximation

$$\frac{dp}{dt}=\omega\approx-\rho gw, \tag{11}$$

Eq. (10) becomes the static energy equation currently in wide use in convective parameterization models, i.e.,

$$\frac{d}{dt}(C_pT+gz)=\dot{Q}. \tag{12}$$

Eq. (11) involves the neglect of $\partial p/\partial t$ (also implicit in Section 3) and the neglect of $\mathbf{V}_h\cdot\nabla_h p$ and the vertical non-hydrostatic pressure field which eliminate the kinetic energy generation terms from Eq. (12). These same approximations are involved in simplifying the well-known parcel kinetic energy equation

$$\frac{d}{dt}(C_pT+gz+\tfrac{1}{2}|\mathbf{V}|^2)=\alpha\frac{\partial p}{\partial t}+\dot{Q}$$

to Eq. (12).

It should be noted that Eq. (12) is not valid for the idealized case of the heating of a static atmospheric column, for which $\omega\equiv0$: this means, for example, that Eq. (12) is not exact if integrated over the globe for an atmosphere not in thermal equilibrium. In general, the validity of Eq. (12) depends on the validity of approximation (11) integrated over the domain of interest. For a few systems where the kinetic energy generation and export is large, such as the hurricane, then the use of the static energy might involve a significant approximation.

5. Conclusion

It appears that if static energy is used as a conserved quantity, the implicit assumption is made that kinetic energy is all locally dissipated; while if poten-tial temperatures are used, one is assuming the maximum available kinetic energy is generated and none is locally dissipated in the chosen atmospheric volume. For many thermal budget calculations (such as for a quasi-steady-state tradewind cumulus layer) it would seem preferable to use static energy, and not include the generation, advection, storage or dissipation of kinetic energy. If potential temperature is used, then the explicit inclusion of the dissipation of kinetic energy (a source term for S) is theoretically necessary, although in practice it is often small. For calculation of model parcel ascents, it would be more appropriate to use potential temperatures (or the equivalent function h_ϕ); but, in practice, this would be inconvenient if static energies are used elsewhere for budget purposes (for example, Arakawa and Schubert, 1974).

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